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ISOMORPHIC TRANSITION FROM COOPERATIVE TO DEFECTIVE VERSIONS OF THE PRISONER'S DILEMMA AND COURNOT DUOPOLY

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Abstract: W pracy pokazano, że gry równoważne ze względu na preferencje indywidualne mogą być nierównoważne z kooperacyjnego punktu widzenia. Zjawisko to występuje zarówno w grach z wyraźną rozbieżnością między postawą rywalizacji uczestników a ich współpracą (dylematy społeczne reprezentowane tu przez dylemat więźnia), jak i w grach konkurencji rynkowej z możliwością tworzenia karteli (oligopole reprezentowane tu przez duopol Cournota). Wprowadzona została definicja izomorfizmu gier celem uściślenia w jakim sensie mówimy o strategicznej równoważności gier. Zyski kooperacyjne obliczane są jako suma wypłat graczy.

Słowa kluczowe: gry izomorficzne, dylemat więźnia, duopol Cournota, użyteczność transferowalna.

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1. Introduction and definitions

The *two-player game* G in a strategic form with strategy sets S_i and payoffs $P_i: S_1 \times S_2 \to \mathbb{R}, i = 1,2,$ is denoted by $G = (S_i, P_i, i = 1,2)$.We write $\mathbb{R}$ for the set of real numbers.

Recall that $(x_1^{*}, x_2^{*}) \in S_1 \times S_2$ is a *Nash equilibrium* of G, if

$$P_i(x_i^{*}, x_{-i}) \leq P_i(x_1^{*}, x_2^{*})$$

for all $x_{-i} \in S_{3-i}$, $i = 1,2$, where (x_2^{*}, x_{-2}) means $(x_{-2}, x_2^{*}) \in S_1 \times S_2$ by standard abuse of notation. A pair of strategies $(\overline{x_1}, \overline{x_2}) \in S_1 \times S_2$ *strongly dominates* (in the sense of Pareto) another pair $(x_1, x_2) \in S_1 \times S_2$, if $P_i(\overline{x_1}, \overline{x_2}) > P_i(x_1, x_2)$ for $i = 1,2$.

A *TU-solution* (TU for transferable utility), is any pair $(\overline{x_1}, \overline{x_2}) \in S_1 \times S_2$ maximizing the sum of payoffs

$$Q(\overline{x_1}, \overline{x_2}) = max_{(x_1, x_2)} Q(x_1, x_2),$$

where $Q = P_1 + P_2$. We treat TU-solution as the cooperative solution of a noncooperative game.

We assume the game to have nontransferable payoffs (NTU), which could pose a problem to applicability of TU-solution. Interestingly in the case of symmeric games (i.e., $P_1(x_1, x_2) = P_2(x_2, x_1)$) this failure of TU can be overcomed and during repetitive play the players need only to alternate between two strategies yielding long-term gains identical with those predicted by fair division of joint TU-payoffs. Thus symmetric NTU games allow for hidden transfer of utility („you make me a favor, then I do you a favor etc."). Moreover, in practical applications of game theory one often allows for explicit side-payments to bridge noncooperative view with cooperative vision, e.g., [3, 4]. We refer to [7, 8, 10] for basics of game theory. Note that we do not assume any topology in strategy sets nor continuity of payoff functions, since the defined objects involve only ordered structure of the real line$\mathbb{R}$.

Following [5] we shall say that two games $G = (S_i, P_i, i = 1,2)$ and $G' = (S_i, P_i', i = 1,2)$(sharing the same strategy sets) are *isomorphic*, if there exist strictly increasing maps $f_i: P_i(S_1 \times S_2) \to \mathbb{R}$, $i = 1,2$, such that

$$P_i'(x_1, x_2) = f_i\left(P_i(x_1, x_2)\right)$$

for $(x_1, x_2) \in S_1 \times S_2$. The symbol $P_i(S_1 \times S_2)$ stands for the subset of $\mathbb{R}$consisting of all payoffs $P_i(x_1, x_2)$ over $(x_1, x_2) \in S_1 \times S_2$.

It can be observed that isomorphic games G and G' possess the same equilibria, Pareto optima etc. (Indeed, strictly increasing map preserves strict order and is invertible from its image). Thus, strategically G and G' are indistinguishable. To put things clearly, the definition of isomorphism accounts only for individual preferences of players who make independent decisions -- a paradigm in noncooperative game theory upon which the concept of Nash equilibrium is build. The aim of this note is to show that *strategically isomorphic games may obey distinct cooperative features*.

Despite the long history of game theory no standard definition of strategic equivalence of games has been proposed so far (cf. chap.4.2 in [8]). Our definition is not stated in a satisfactory generality, but sounds naturally and it can be easily employed. We believe that any future definition of isomorphism for games, based primarily on individual preferences of players, would contain our definition as a particular case.

To keep our presentation concise and avoid notational difficulties we restricted the discussion to two-player games. The extension of the notion of isomorphism (as understood in this article) and its strategic properties to multiplayer games is merely a technical issue.

2. From cooperative to defective prisoner's dilemma

The *prisoner's dilemma* $PD(a, b, c, d)$ is defined to be the game with strategy sets $S_1 = S_2 = \{C, B\}$ (C -- cooperate, B -- betray), and payoffs $[P_1(x_1, x_2), P_2(x_1, x_2)]$, $(x_1, x_2) \in S_1 \times S_2$, given by the following bimatrix (Table1).

Table 1. Payoffs in prisoner's dilemma ($a < b < c < d$)

	C	B
C	$[c, c]$	$[a, d]$
B	$[d, a]$	$[b, b]$

A pair (B, B) is a unique Nash equilibrium of PD and (C, C) is a Pareto optimum dominating (B, B); two other optima are (B, C) and (C, B). This invites the famous paradox that players committed to play (C, C), strategically unsafe option, fare better than those staying in equilibrium (B, B).

All games $PD(a, b, c, d)$ with $a < b < c < d$ are isomorphic, so one encounters various concrete values of a, b, c, d in the literature, which does not change the social dilemma. However, if we look closer at the cooperative structure of $PD(a, b, c, d)$, namely when the game is played iteratively, an additional condition is assumed to prevent quirky behavior of players: $c > (a + d)/2$, see p.345 in [1], Notes to chap.1 in [7] or [6]. If $c > (a + d)/2$, then cooperation (C, C) strongly dominates Nash equilibrium. However, if $c < (a + d)/2$, then it is beneficiary to play alternately $\ldots, (C, B), (B, C), \ldots$. Roughly speaking, an arrangement of precommitted betrayal becomes feasible in a long-term perspective to such extent that it is even more attractive than simple cooperation. Moreover, fake betrayal realizes the premise of TU-solution, although the game itself is NTU.

Therefore we observe the qualitative transition from cooperative to defective versions of PD. For critical value $c = (a + d)/2$ both kinds of behavior: cooperative (C, C) and defective (C, B) and (B, C) alternately, lead to equally profitable outcomes.

3. From cooperative to defective Cournot duopoly

The *Cournot duopoly* $CD(k)$ is defined here to be the game with strategy sets $S_1 = S_2 = [0, L]$, $L > 0$, and payoffs

$$P_i(x_1, x_2) = (max\{0, x_i \cdot (L - x_1 - x_2)\})^k,$$

$x_i \in S_i$, $i = 1,2$, $k > 0$; cf. [2, 7, 10]. A pair $(L/3, L/3)$ is a unique Nash equilibrium of CD.

It is known that both players/firms would fare better by committing to (form a cartel and) play/produce $(L/4, L/4)$. Aside of illegality of such behavior of oligopolists (e.g., [9]), it is also strategically unjustified for one-shot games, so we have to recourse to repeated/dynamic games to explain the phenomenon of cartel within the game theoretic framework (cf. [6]).

All games $CD(k)$ are isomorphic. (In particular it is enough to compute equilibria and optima in the simplest case $k = 1$). Nevertheless they differ with respect to cooperation quantified by TU-solution.

The set of maximizers of $Q = P_1 + P_2$ comprises of all TU-solutions and it can be established by elementary multivariable calculus. For $k = 1$ we find that TU-solutions form the interval $x_1 + x_2 = L/2$, $x_1, x_2 \in [0, L]$. Further, for $k \neq 1$, Q possesses on the square $[0, L] \times [0, L]$ either interior maximum at $(L/4, L/4)$ with value $Q(L/4, L/4) = 2 \cdot (L/4)^k \cdot (L/2)^k$, or two boundary maxima at $(L/2,0)$ and $(0, L/2)$ with value $Q(L/2,0) = Q(0, L/2) = (L/2)^k \cdot (L/2)^k$. The first option is realized for $k > 0$ satisfying $2 > 2^k$, i.e., $0 < k < 1$. The second option is realized for k satisfying $2 < 2^k$, i.e., $k > 1$.

Summarizing, $CD(k)$ for $k < 1$ has cooperative flavor: it is good to keep suitably adjusted identical production levels (lower than at equilibrium), and for $k > 1$ the duopoly has defective flavor: it is good to make an arrangement between firms so that only one firm keeps an optimal production level and the other firm rests with no production; of course for hidden transfer of utility the roles of producers ought be switched. As the parameter k changes from 0 toward $+\infty$ the family of isomorphic games $CD(k)$ undergoes the qualitative transition from cooperative to defective version, see Figure 1 for the corresponding bifurcation diagram.

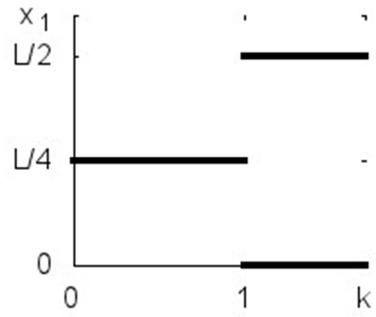

Fig. 1 Transition from TU-solution $(L/4, L/4)$ to a pair of TU-solutions $(L/2,0)$, $(0, L/2)$ in $CD(k)$ as k varies.

Finally, let us observe that from the Cournot duopoly one obtains the prisoner's dilemma $PD(a, b, c, d)$ by putting $C := L/4$, $B := L/3$ for strategies and $a := P_1(L/4, L/3) = P_2(L/3, L/4)$, $b := P_i(L/3, L/3)$, $c := P_i(L/4, L/4)$, $d := P_2(L/4, L/3) = P_1(L/3, L/4)$, $i = 1,2$, in the payoff bimatrix.

4. Summary

We showed that games which are equivalent with respect to individual preferences can be nonequivalent with respect to cooperative gains. This phenomenon is present in games with strong competitive vs cooperative tension (social dilemmas, exemplified here by the prisoner's dilemma) as well as games of market concurrency with prospect for cartel creation (oligopolies, exemplified here by the Cournot duopoly). To make precise meaning of strategically equivalent games we introduced the notion of isomorphism of games. Cooperative gains are calculated as sums of payoffs of players.

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